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Suresh et al.

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(54) **SINGULARITY REDUCTION IN QUADRILATERAL MESHES**

(71) Applicant: **Wisconsin Alumni Research Foundation**, Madison, WI (US)

(72) Inventors: **Krishnan Suresh**, Middleton, WI (US); **Chaman S. Verma**, Madison, WI (US)

(73) Assignee: **Wisconsin Alumni Research Foundation**, Madison, WI (US)

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G06T 19/00	(2011.01)
G06T 7/13	(2017.01)

(52) **U.S. Cl.**
CPC ***G06T 17/205*** (2013.01); ***G06T 7/13*** (2017.01); ***G06T 19/00*** (2013.01); *G06T 2200/04* (2013.01); *G06T 2207/20021* (2013.01)

(58) **Field of Classification Search**
None
See application file for complete search history.

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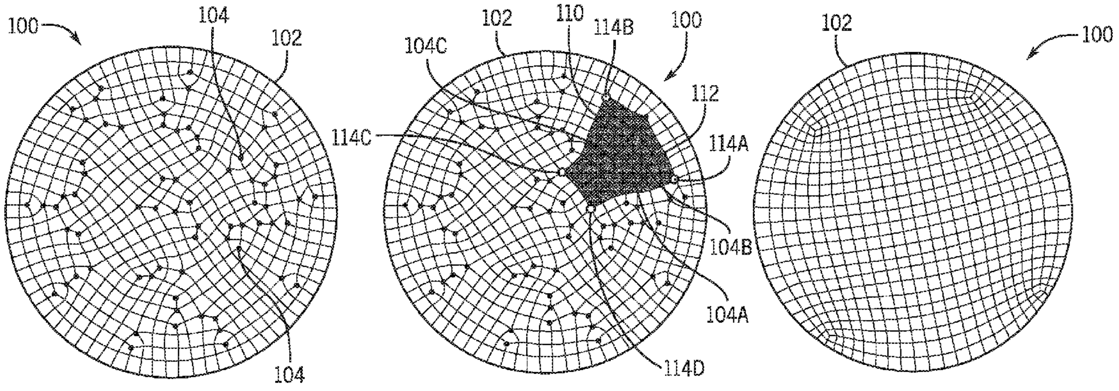
Primary Examiner — Kee M Tung
Assistant Examiner — Patrick F Valdez
(74) *Attorney, Agent, or Firm* — Quarles & Brady LLP

(57) **ABSTRACT**

Systems and methods for modifying and generating quadrilateral meshes for computer graphic structures include obtaining a polygon mesh representing a computer graphic structure, the polygon mesh comprising a plurality of polygonal faces and a plurality of singularities, determining, based on a first singularity of the plurality of vertices, selecting, based on one or more characteristics of the patch, a first minimum singularity template (MST) of a plurality of MSTs each representing a corresponding quadmesh that has three or fewer singularities, and replacing, within the polygon mesh, the patch with the first MST.

20 Claims, 16 Drawing Sheets



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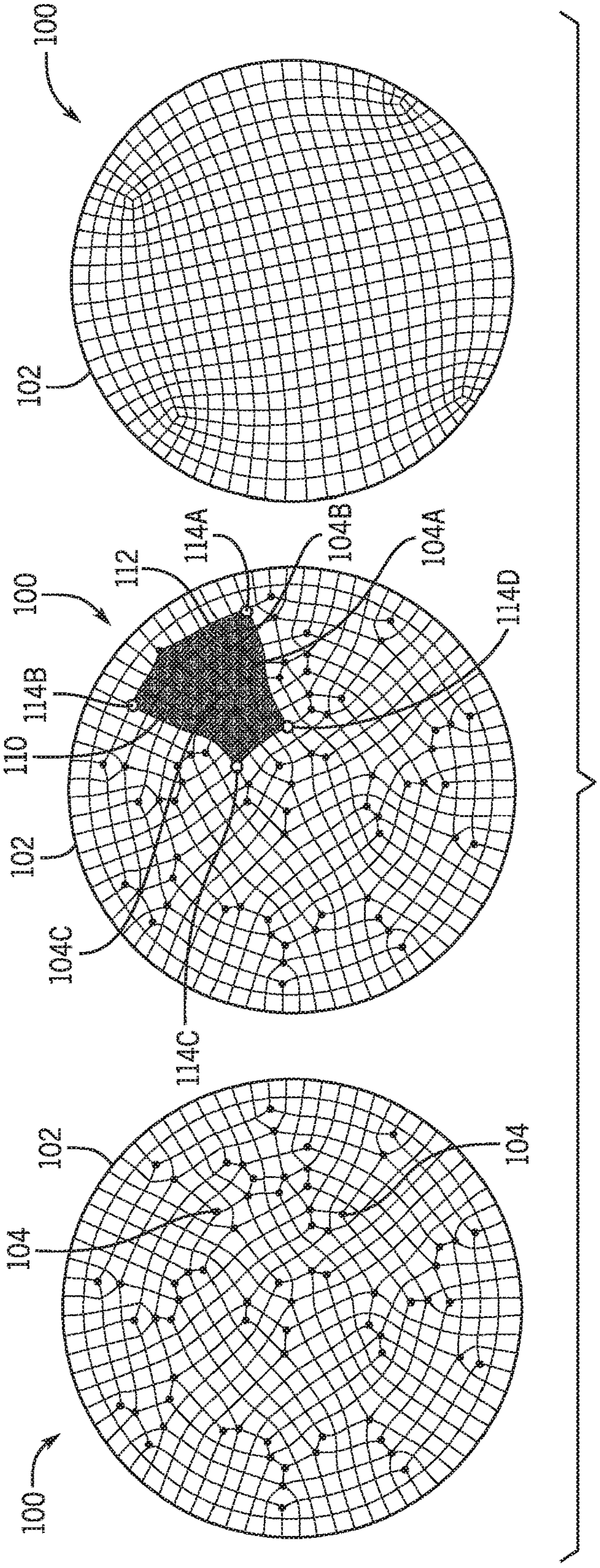

100
102
100
112
114A
104B
104A
114D
114B
110
102
104C
114C
102
104
104
100

FIG. 1

FIG. 2A

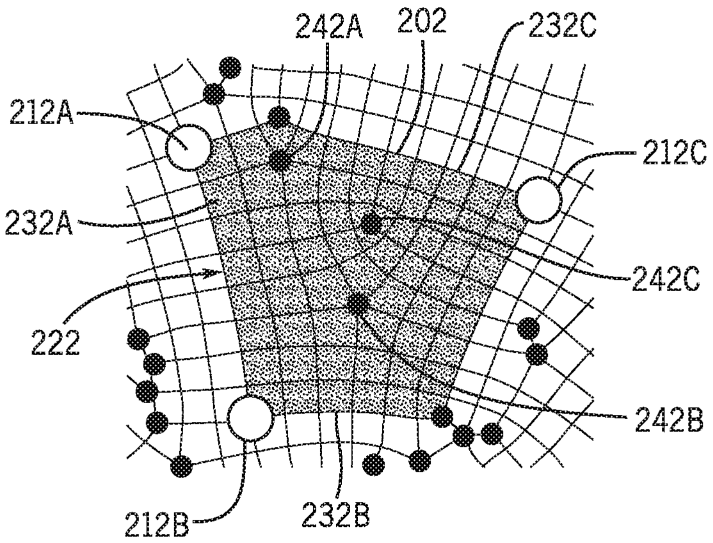


FIG. 2B

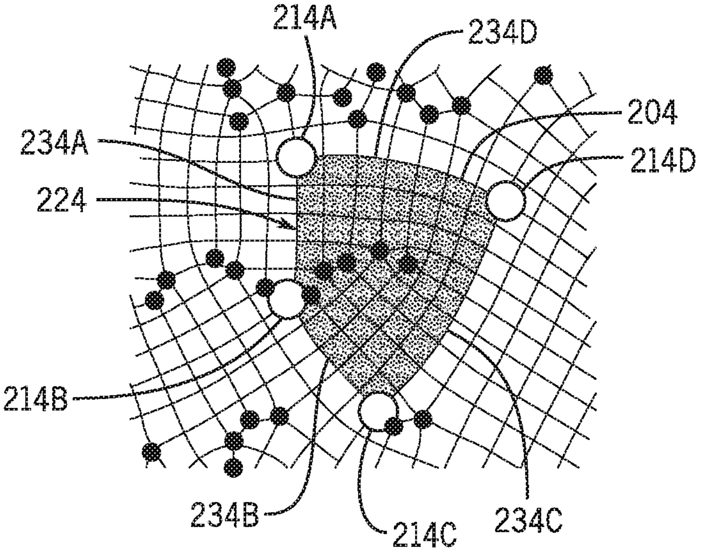
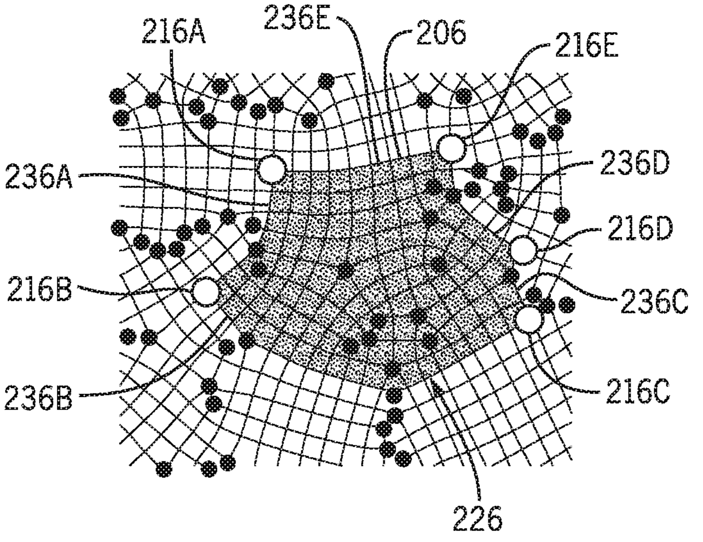


FIG. 2C



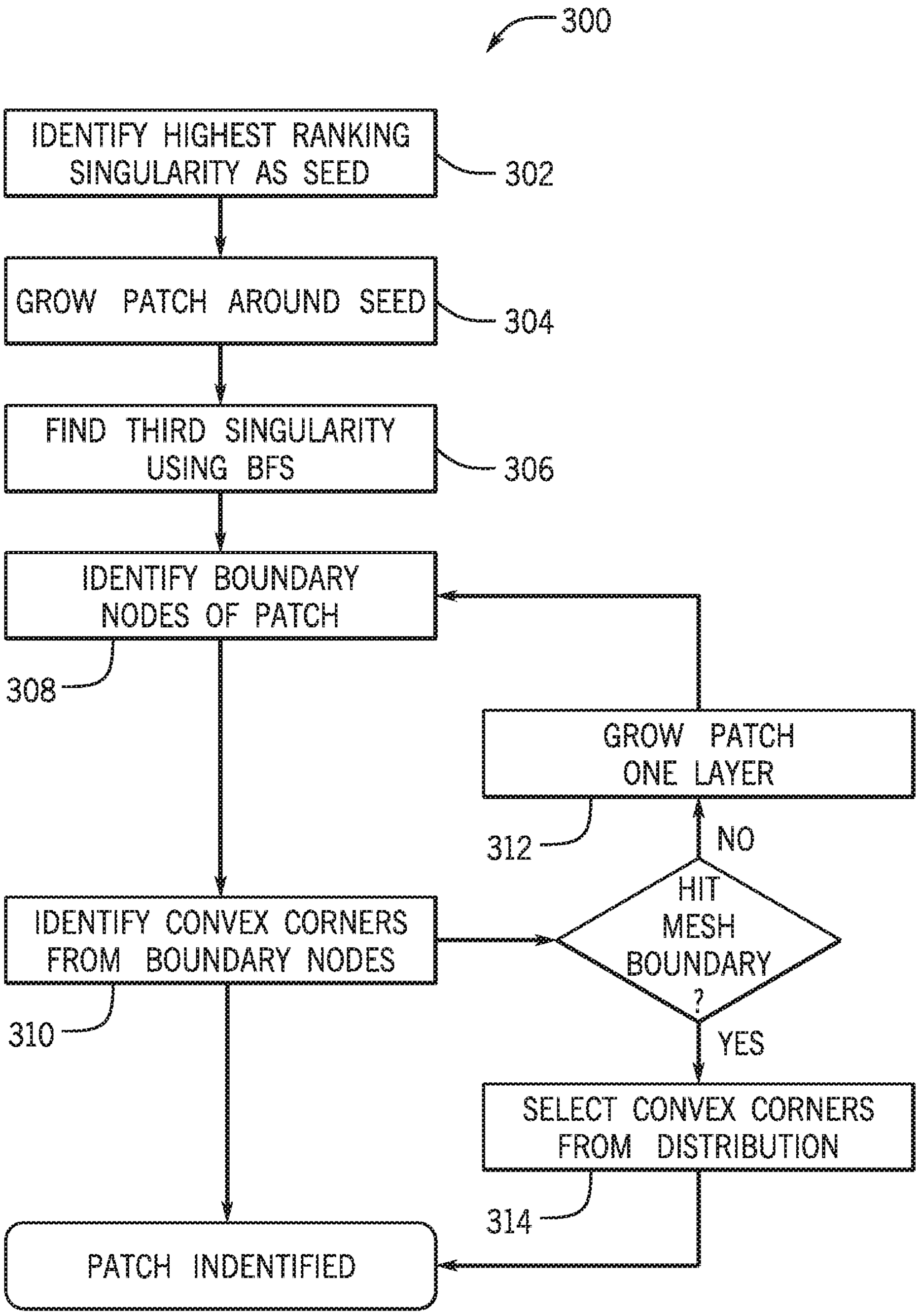

300
IDENTIFY HIGHEST RANKING SINGULARITY AS SEED
302
GROW PATCH AROUND SEED
304
FIND THIRD SINGULARITY USING BFS
306
IDENTIFY BOUNDARY NODES OF PATCH
308
GROW PATCH ONE LAYER
312
NO
HIT MESH BOUNDARY ?
IDENTIFY CONVEX CORNERS FROM BOUNDARY NODES
310
YES
SELECT CONVEX CORNERS FROM DISTRIBUTION
314
PATCH INDENTIFIED

FIG. 3

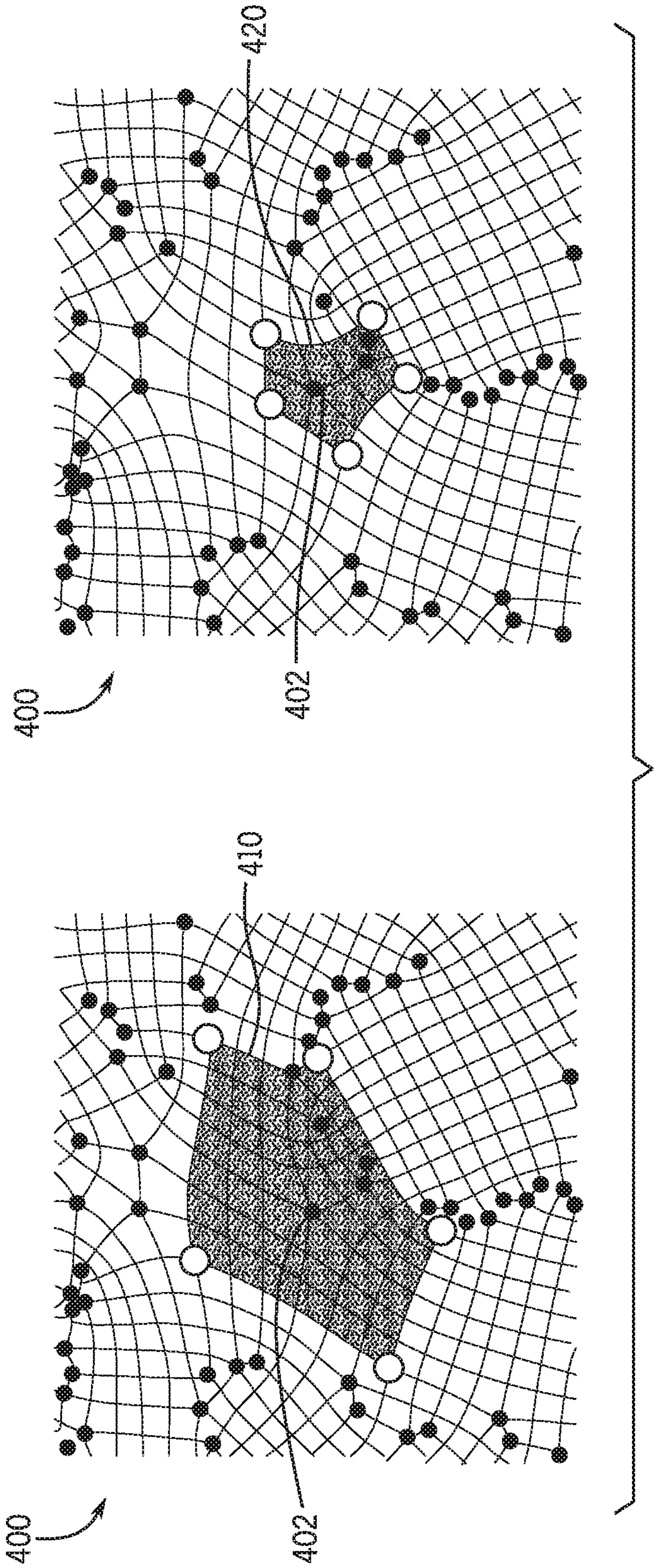


FIG. 4

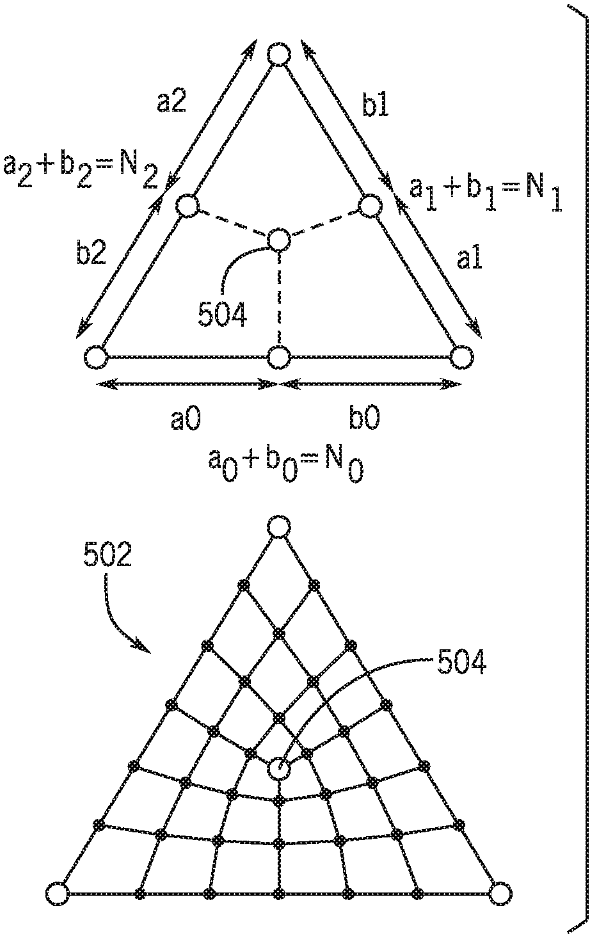

a2
b1
$a_2+b_2=N_2$
$a_1+b_1=N_1$
b2
a1
504
a0
b0
$a_0+b_0=N_0$
502
504

FIG. 5A

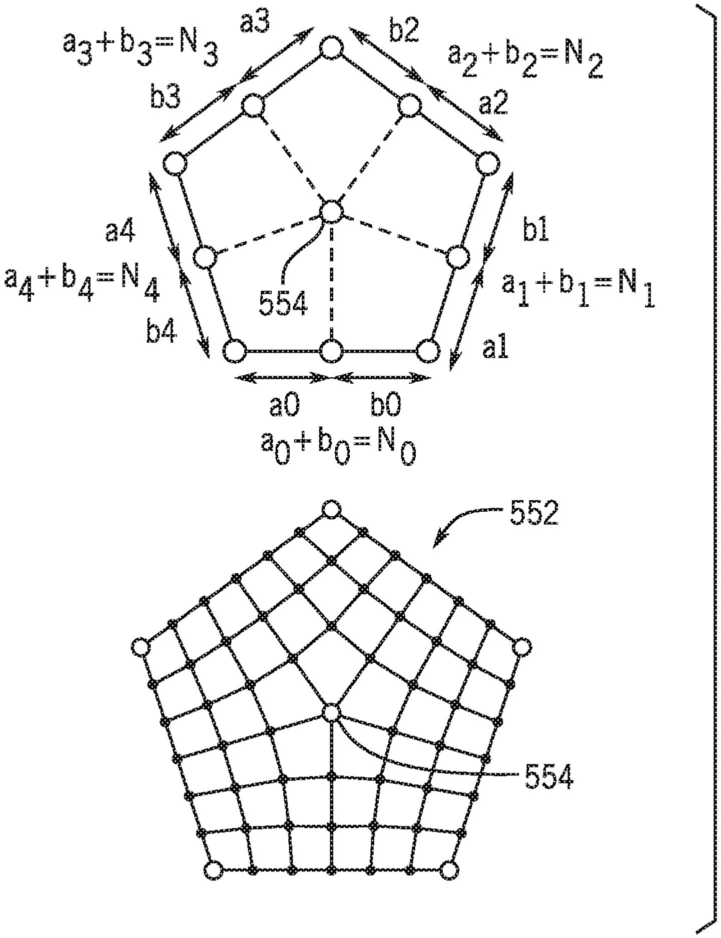

a3
b2
$a_3+b_3=N_3$
$a_2+b_2=N_2$
b3
a2
a4
b1
$a_4+b_4=N_4$
554
$a_1+b_1=N_1$
b4
a1
a0
b0
$a_0+b_0=N_0$
552
554

FIG. 5B

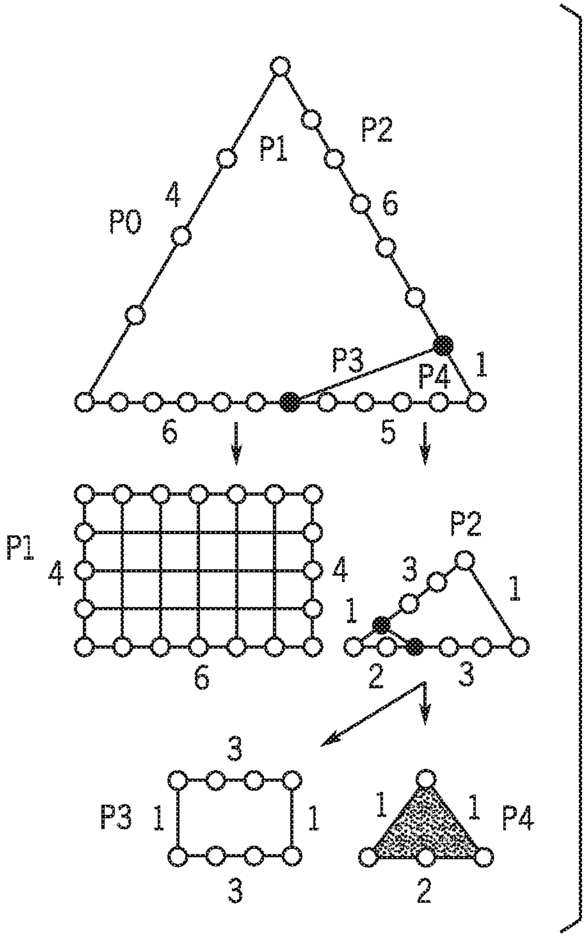
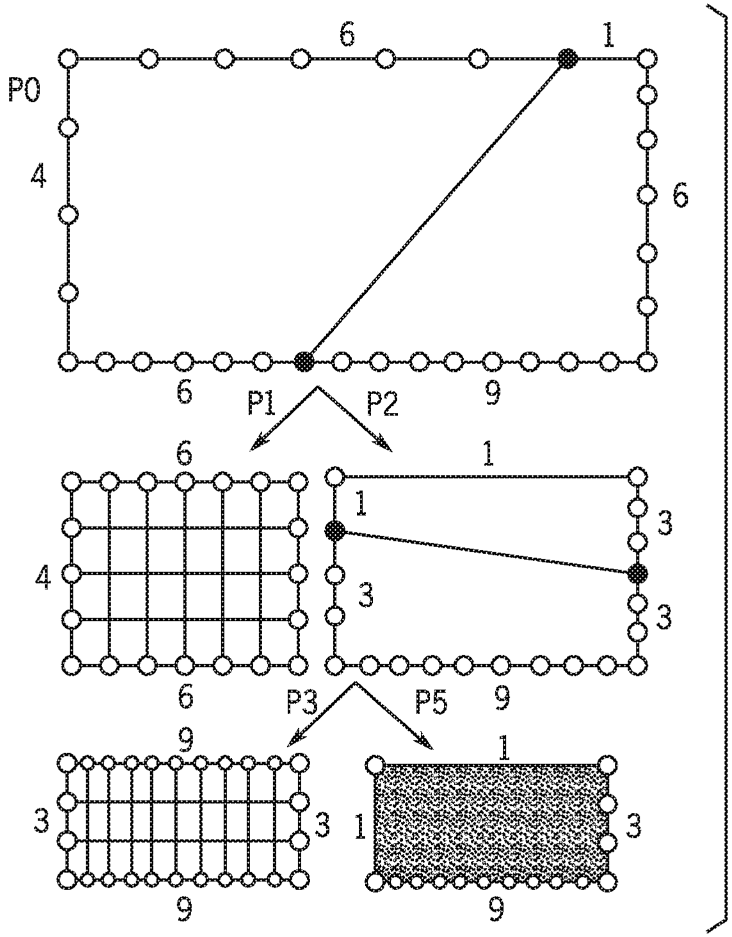

P0
4
P1
P2
6
P3
P4
1
6
5
P1
4
4
6
P2
3
1
1
2
3
3
P3
1
1
3
1
1
P4
2

FIG. 6A


6
1
P0
4
6
6
9
P1
P2
6
1
1
3
4
3
3
6
9
P3
P5
9
1
3
3
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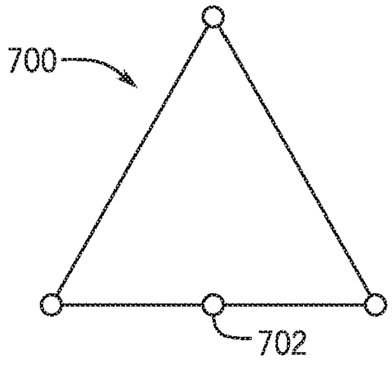
3
9
9

FIG. 6B

FIG. 7A



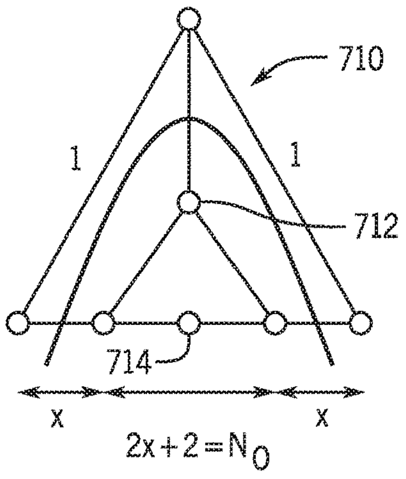


FIG. 7B

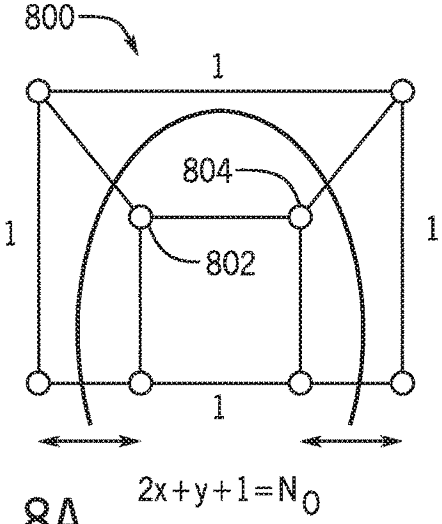


FIG. 8A

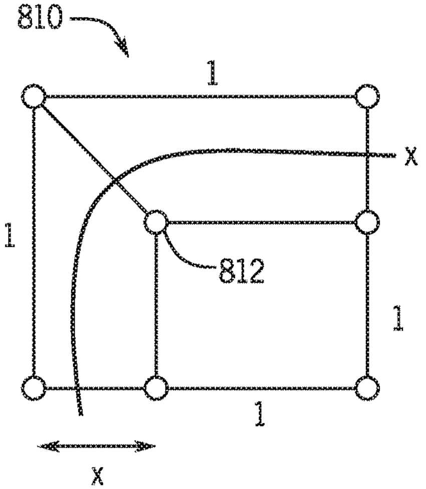
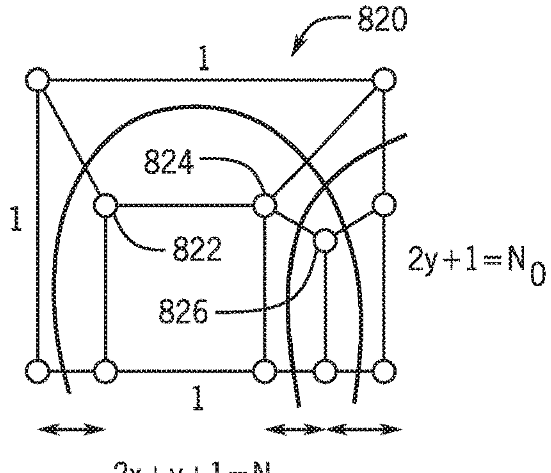


FIG. 8B



$2x+y+1=N_O$

FIG. 8C

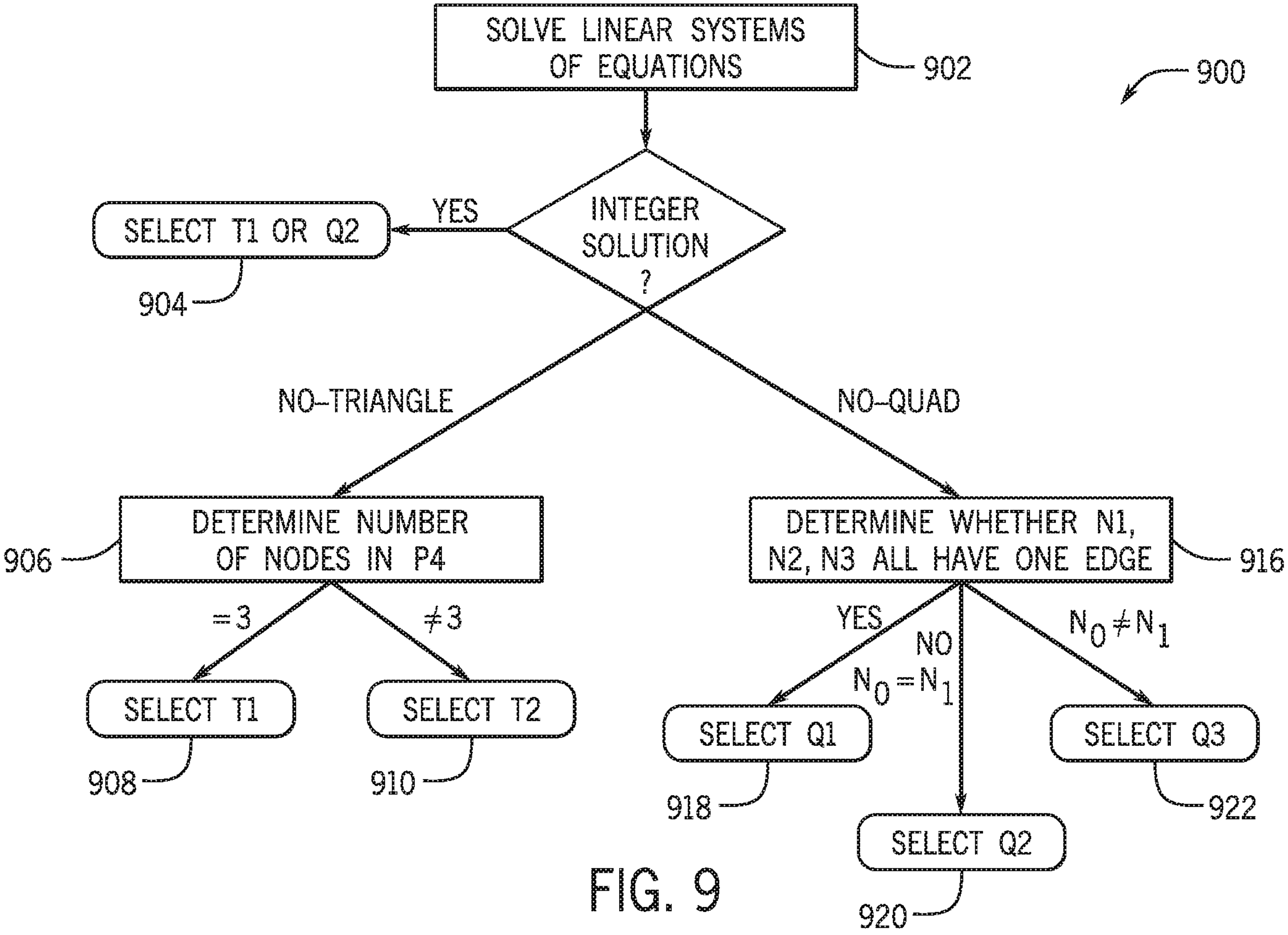

900
SOLVE LINEAR SYSTEMS OF EQUATIONS
902
INTEGER SOLUTION ?
YES
SELECT T1 OR Q2
904
NO–TRIANGLE
NO–QUAD
DETERMINE NUMBER OF NODES IN P4
906
=3
≠3
SELECT T1
908
SELECT T2
910
DETERMINE WHETHER N1, N2, N3 ALL HAVE ONE EDGE
916
YES
NO
N0=N1
N0≠N1
SELECT Q1
918
SELECT Q2
920
SELECT Q3
922

FIG. 9

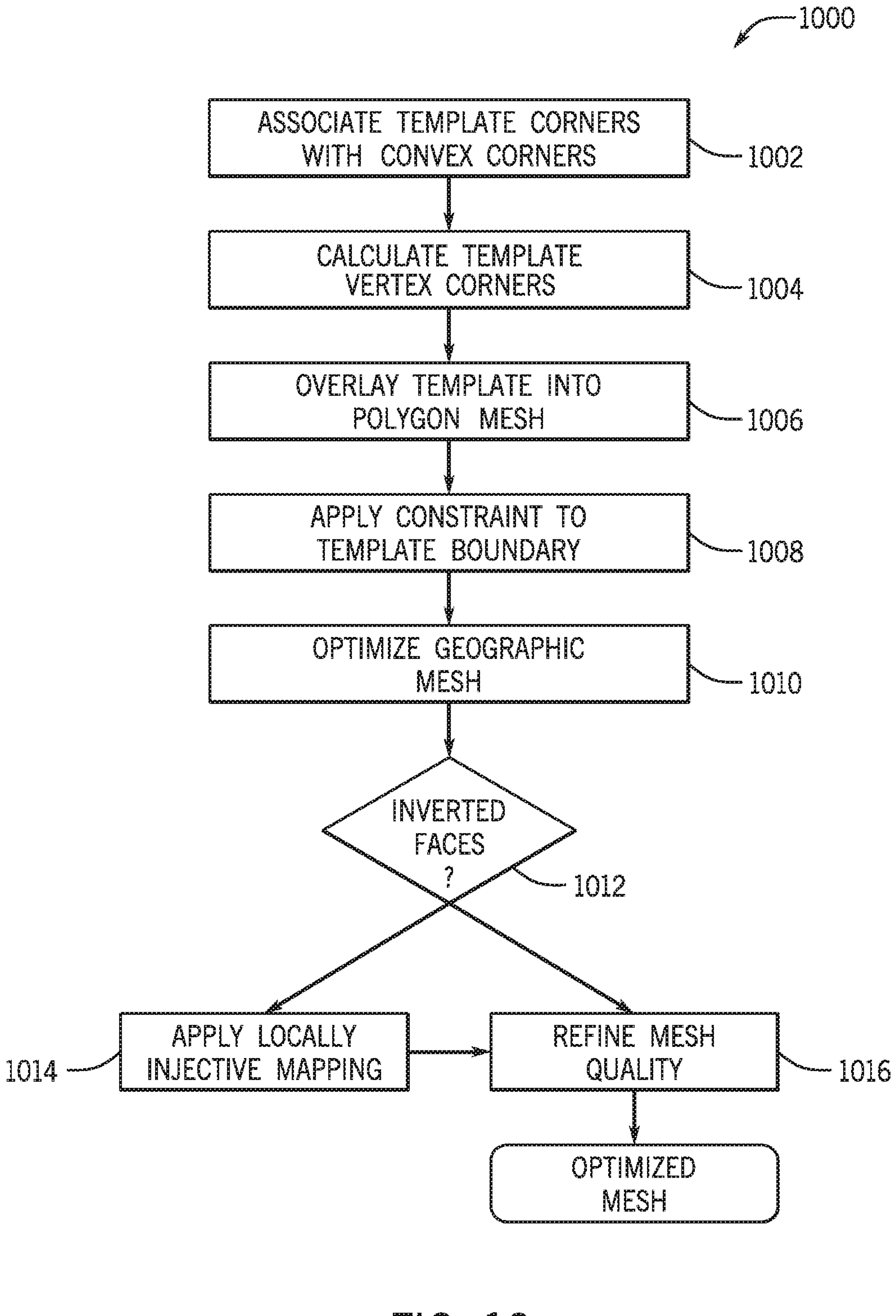

1000
ASSOCIATE TEMPLATE CORNERS WITH CONVEX CORNERS
1002
CALCULATE TEMPLATE VERTEX CORNERS
1004
OVERLAY TEMPLATE INTO POLYGON MESH
1006
APPLY CONSTRAINT TO TEMPLATE BOUNDARY
1008
OPTIMIZE GEOGRAPHIC MESH
1010
INVERTED FACES ?
1012
1014
APPLY LOCALLY INJECTIVE MAPPING
REFINE MESH QUALITY
1016
OPTIMIZED MESH

FIG. 10

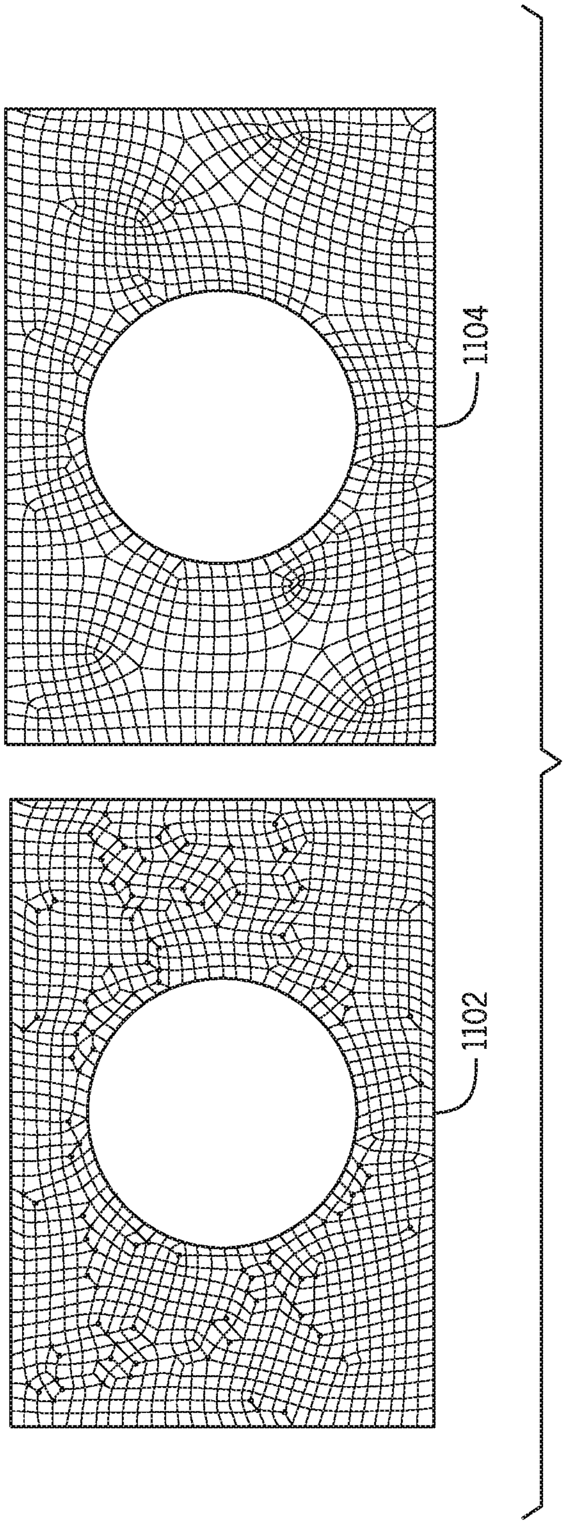


FIG. 11A

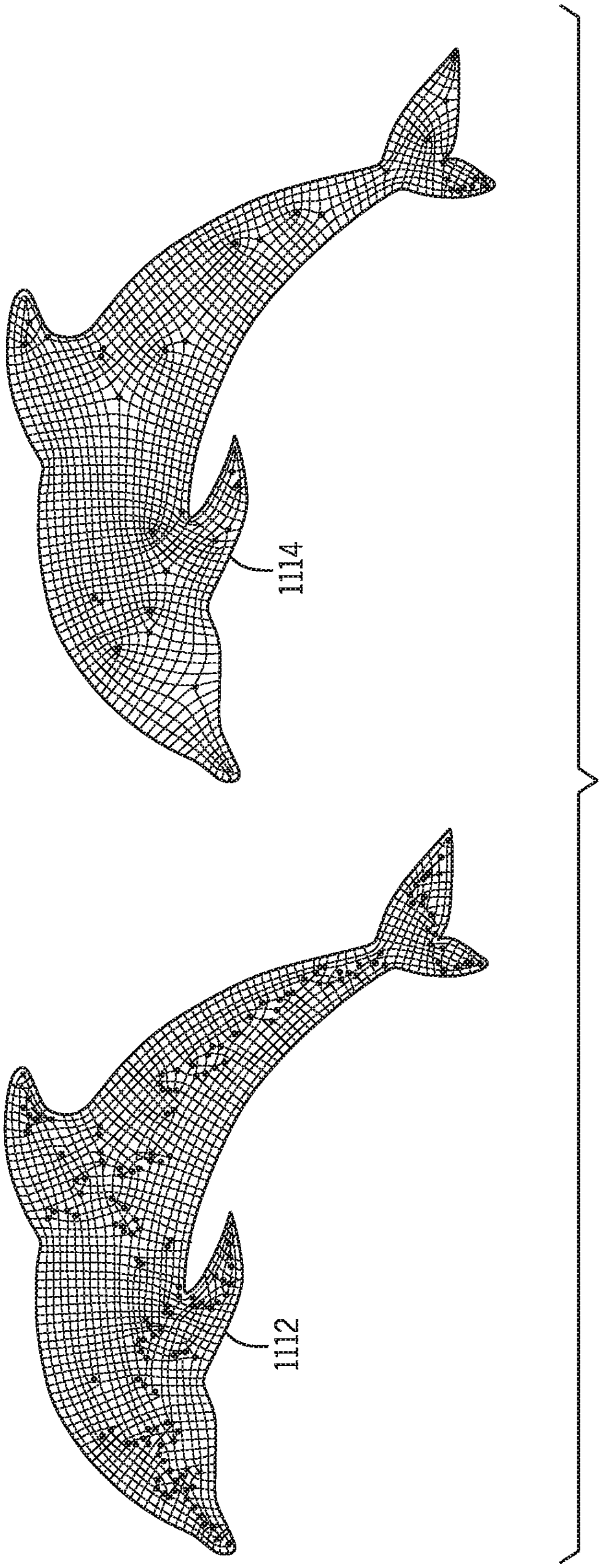


FIG. 11B

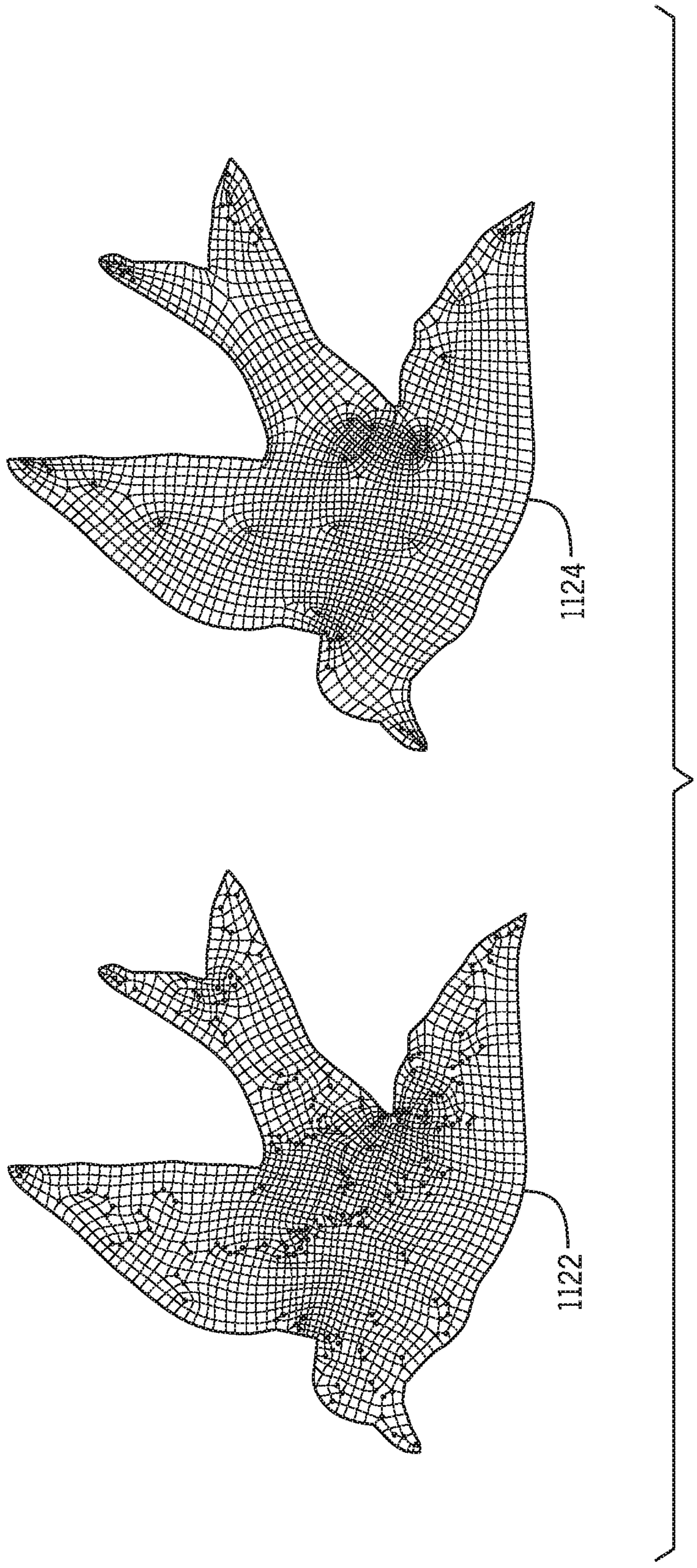


FIG. 11C

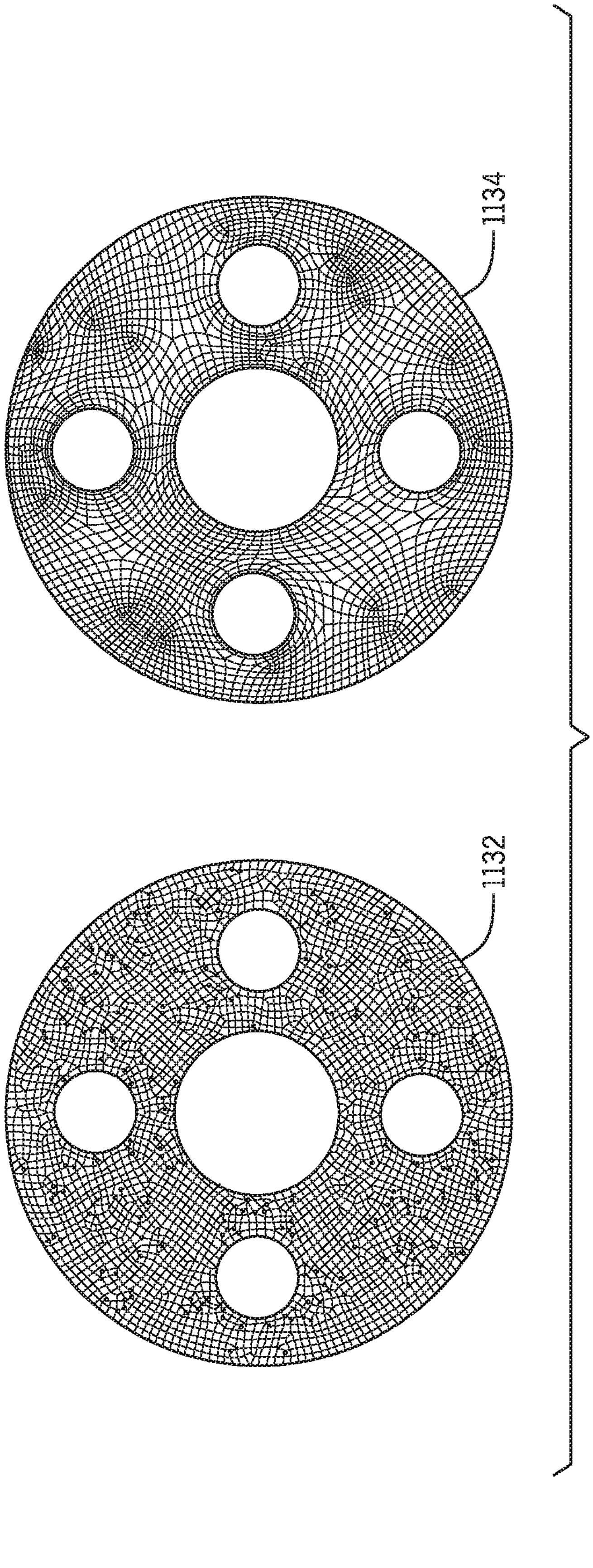


FIG. 11D

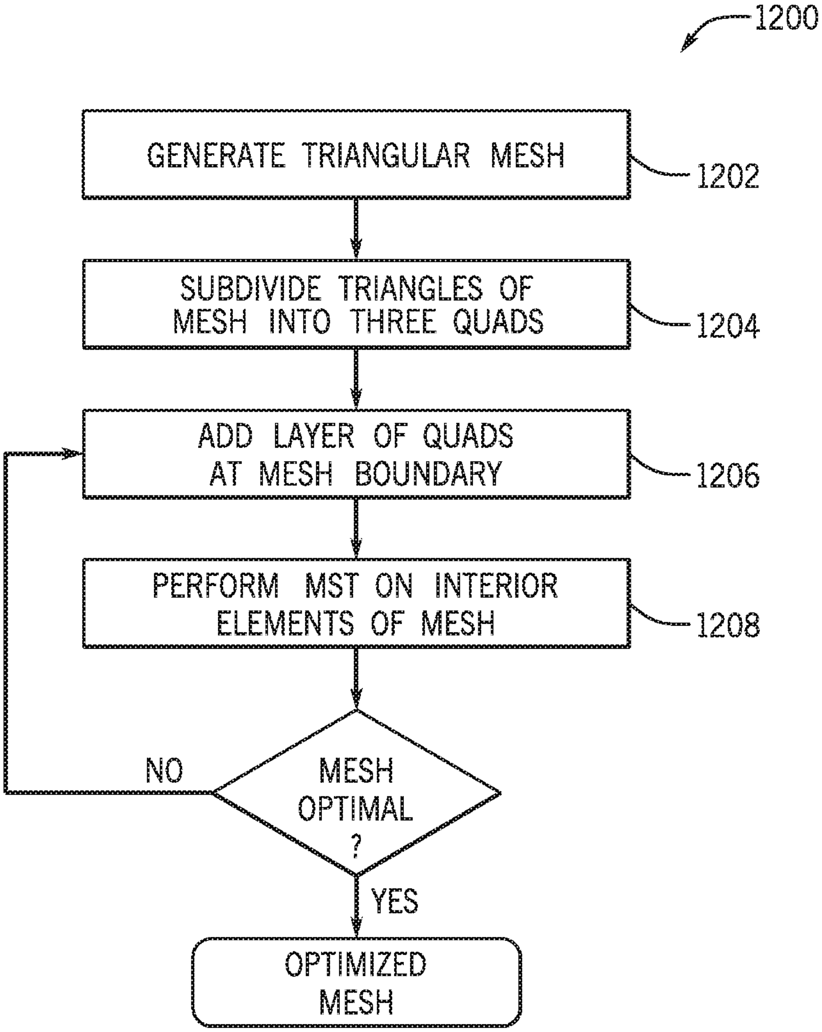

1200
GENERATE TRIANGULAR MESH
1202
SUBDIVIDE TRIANGLES OF MESH INTO THREE QUADS
1204
ADD LAYER OF QUADS AT MESH BOUNDARY
1206
PERFORM MST ON INTERIOR ELEMENTS OF MESH
1208
MESH OPTIMAL ?
NO
YES
OPTIMIZED MESH

FIG. 12

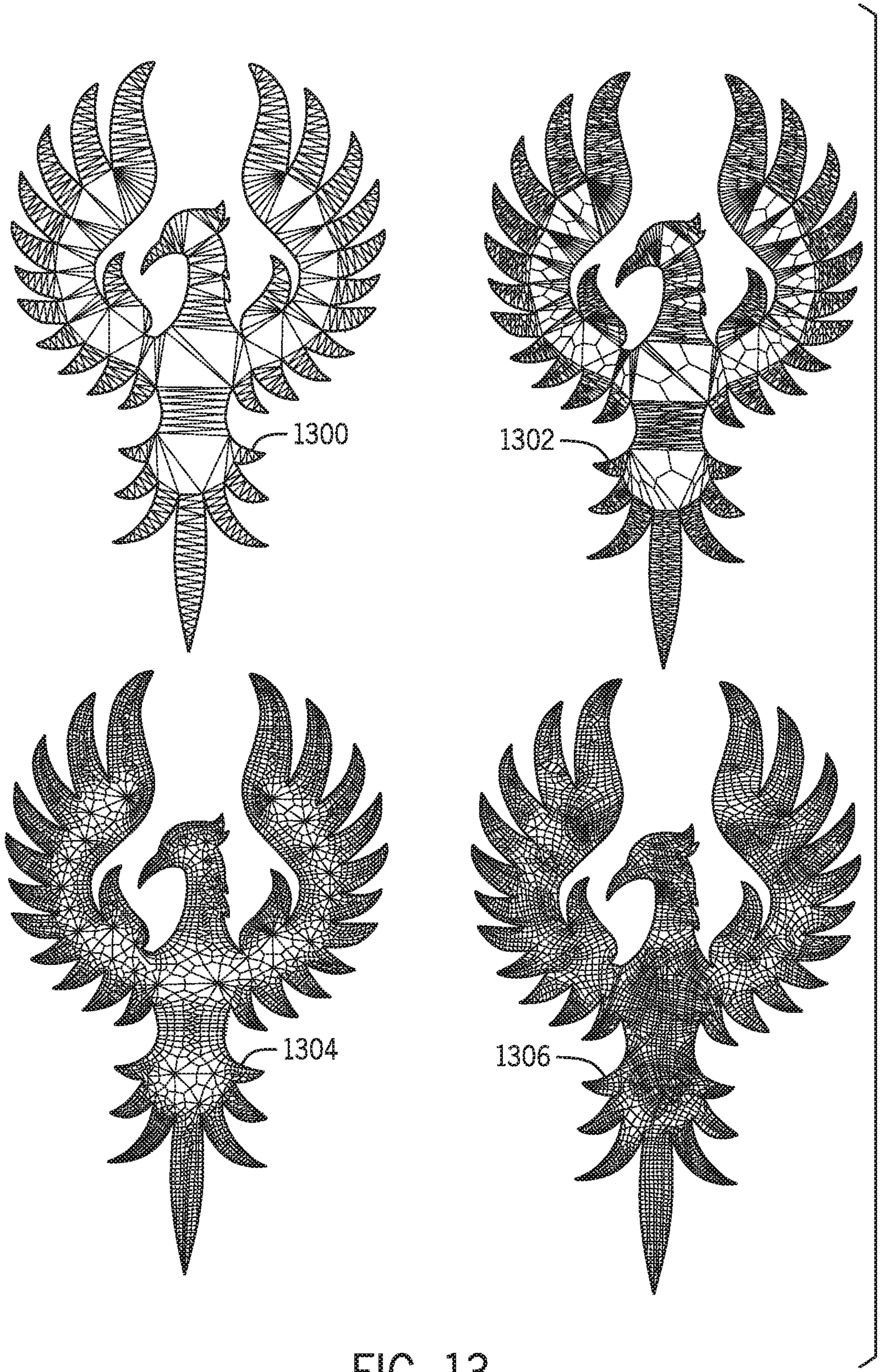


FIG. 13

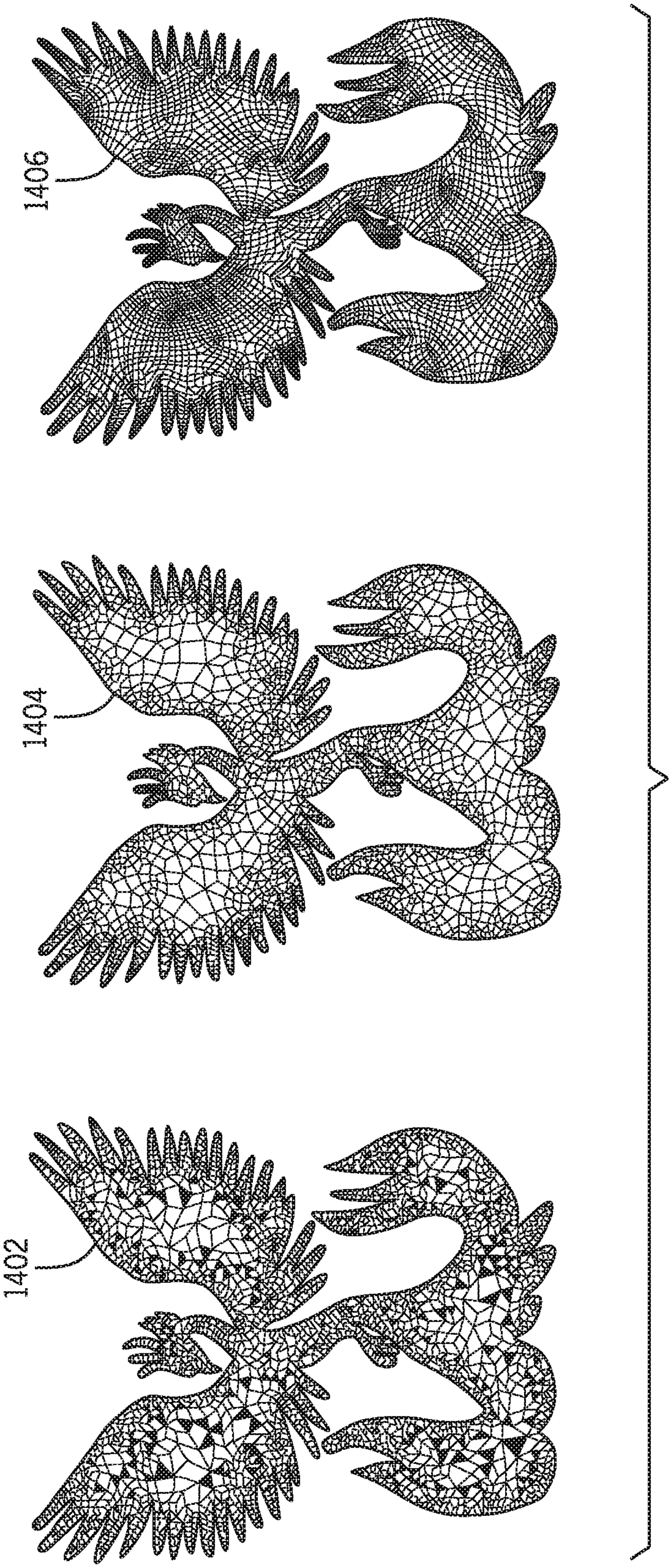


FIG. 14

SINGULARITY REDUCTION IN QUADRILATERAL MESHES

STATEMENT REGARDING FEDERALLY SPONSORED RESEARCH

This invention was made with government support under CMMI1161474 awarded by the National Science Foundation. The government has certain rights in the invention.

BACKGROUND OF THE INVENTION

This disclosure relates generally to methods of optimizing meshes in computer simulations, and more specifically to methods for reducing singularities in quadrilateral meshes.

Use of computer aided design (CAD) tools is common in the engineering industry and also finds use in theatrical animation and graphics. One critical component of CAD tools is the ability to create a three-dimensional collection of polygons, called a "mesh," over a complex structure so that further analysis of the structure can be accomplished. That is, subsequent computing operations may be performed using the mesh as a substitute for the structure.

In 2D, the polygons in a mesh can be any combination of triangles, quadrilaterals, and polygons with five or more sides. In 3D, the mesh can be a combination of tetrahedral and hexahedral elements. There are many applications, especially in non-linear structural mechanics, linear elastic simulations, higher order spectral methods, and texture mapping, which are sensitive to the directions, curvatures, and features on the geometric models. In such applications, an all-quadrilateral and all-hexahedral mesh is often preferred over a triangular mesh. A good quadrilateral mesh is characterized by both topological and geometric quality metrics. Geometric quality is measured from metrics such as aspect ratio, min/max angles, area, etc. Topological quality is measured by regularity in vertex distribution. An internal vertex is considered "regular" if it has four incident edges, otherwise it is a "singular" node, or a "singularity."

The Gauss-Bonnet theorem states that all surfaces with positive genus must have singularities. In addition, singularities are essential in controlling distortions near bifurcations, protrusions, cavities etc. and in abrupt shape transitions. However, singularities in mesh could lead to (1) numerical instability in computational fluid dynamics (CFD) applications (2) wrinkles in subdivision surfaces, (3) irrecoverable element inversions near concave boundaries, (4) helical patterns, (5) visible seams in texture maps, and (6) breakdown of structured patterns on manifolds. Therefore, the major challenges in producing high-quality quad and hex mesh generators are usually related to minimization and placement of singularities.

There are several automatic mesh generators that can produce a high geometric quality mesh. However, the resulting mesh is typically of low topological quality—specifically, the mesh is not optimized with respect to singularities—which is detrimental in the downstream analysis or graphics applications. There are no known techniques to create meshes with both high geometric and high topological qualities, but having such a mesh is critical to producing the fastest, and most accurate, finite element analysis (FEA). Similarly, there are several techniques today that assist in mesh improvement, mesh refinement and mesh simplification. However, while these yield meshes with acceptable geometric quality, they provide very limited control over topological quality.

Further, quad and hex meshes are inherently global in nature: a single modification to their topology can have a domino effect, in that a large number of elements may have to undergo modifications to keep the mesh consistent. The non-localness compounds difficulties in the automation of quad mesh generation and editing, as various quality criteria have non-linear dependencies which can be extremely hard to encode and solve. Furthermore, it is also impossible to refine, coarsen, or edit a quad mesh with local operations, as they create additional singularities. For all these reasons, quad and hex meshing problems are often formulated as global optimization problems. Unfortunately, these optimizations are expensive, and their parametric tweaking is non-intuitive. There is no direct intuitive connection between the user constraints and the resulting mesh topology.

While there have been some significant developments in automatic quadrilateral mesh generation, a robust framework for addressing both geometric and topological quality is needed. Therefore, the purpose of this disclosure is to provide a methodology for reducing singularities in a quadrilateral mesh without loss of geometric quality.

SUMMARY OF THE INVENTION

The present invention overcomes the aforementioned drawbacks by providing a simple and robust method to reduce singularities in a given quad mesh. The main idea is to replace sub-meshes containing a large number of singularities with Minimum Singularity Templates. This process is applied repeatedly while maintaining desired geometric quality.

In one aspect, the disclosure provides a method of optimizing a computer graphic structure. The method includes: receiving a polygon mesh associated with the structure, the polygon mesh comprising a plurality of polygonal faces and a plurality of vertices each located at an intersection of at least two of the faces, the plurality of vertices including a plurality of singularities at which more or less than four of the faces intersect; receiving a criterion for ranking the plurality of singularities; determining, using the criterion, that a first singularity of the plurality of singularities is the highest ranking singularity; identifying, using a shortest path algorithm, that a second singularity of the plurality of singularities is topologically closest, of the plurality of singularities within the polygon mesh, to the first singularity; identifying, using a breadth-first search of the plurality of vertices that begins at a first vertex disposed between the first singularity and the second singularity, a third singularity of the plurality of singularities; determining a patch of the polygon mesh, the patch containing the first singularity, the second singularity, and the third singularity and having a first boundary formed by a plurality of sides each defined by one or more of a plurality of edges between two of the plurality of faces, and a plurality of convex corners each defined by a corresponding vertex of the plurality of vertices, each of the convex corners located at the intersection of two of the plurality of sides; determining a first minimum number of singularities that must be located within the patch; selecting, based on the first minimum number of singularities, a first minimum singularity template (MST) of a plurality of MSTs each representing a corresponding quadmesh; and replacing, within the polygon mesh, the patch with the first MST.

Determining the first minimum number of singularities for the patch may include: dividing the patch into a plurality of sub-patches, wherein a first sub-patch of the plurality of sub-patches is a maximally reduced patch (MRP) having a second boundary with an even number of the plurality of edges and a first MRP side defined by exactly one of the plurality of edges, and each other of the plurality of sub-patches are quadrilateral and each contain a corresponding quadmesh that has no singularities; and determining, as the first minimum number of singularities for the patch, a second minimum number of singularities that must be located within the first sub-patch. Selecting the first MST may include determining that the patch is quadrangulable with only one singularity. Selecting the first MST may alternatively include determining whether the patch is triangular or quadrilateral and selecting the first MST from the plurality of MSTs comprising: a first triangular MST for a triangular patch, the first triangular MST having one internal singularity; a second triangular MST for the triangular patch, the second triangular MST having two internal singularities; a first quadrilateral MST for a quadrilateral patch, the first quadrilateral MST having two internal singularities; a second quadrilateral MST for the quadrilateral patch, the second quadrilateral MST having one internal singularity; and, a third quadrilateral MST for the quadrilateral patch, the third quadrilateral MST having three internal singularities. Replacing the patch with the first MST may include: creating a correspondence between each corner of a plurality of corners of the MST with a corresponding convex corner of the plurality of convex corners of the patch; calculating a set of coordinates for each quadmesh vertex of a plurality of quadmesh vertices of the first MST; overlaying, using the correspondence between corners and the sets of coordinates, the quadmesh of the first MST onto the polygon mesh; and applying a locally injective mapping to each inverted face of a plurality of faces of the quadmesh to produce a fold-free mesh in which each of the plurality of faces has a positive Jacobian.

In another aspect, the present disclosure provides a method that includes: obtaining a polygon mesh representing a computer graphic structure, the polygon mesh comprising a plurality of polygonal faces and a plurality of vertices each located at an intersection of at least two of the faces, wherein some of the plurality of vertices are singularities; determining, based on a first singularity of the plurality of vertices, a patch of the polygon mesh, the patch containing the first singularity and having a first boundary; selecting, based on one or more characteristics of the patch, a first minimum singularity template (MST) of a plurality of MSTs each representing a corresponding quadmesh that has three or fewer singularities; and replacing, within the polygon mesh, the patch with the first MST. Obtaining the polygon mesh may include: generating a triangular mesh for the structure; subdividing each triangle in the triangular mesh into three quadrilaterals to produce a quadmesh with no singularities; and adding one or more layers of new quadrilaterals into the quadmesh, thus producing the vertices that are singularities.

Determining the patch may include identifying, as a second singularity, the vertex of the plurality of vertices that is the topologically closest singularity to the first singularity, the patch containing the second singularity. Identifying the second singularity may include performing a Dijkstra shortest distance search from the first singularity. Determining the patch may further include identifying, using a breadth-first search of the plurality of vertices that begins at a first vertex disposed between the first singularity and the second singularity, a third singularity of the plurality of vertices, the patch containing the third singularity. Selecting the first MST may include dividing the patch into a plurality of sub-patches, wherein a first sub-patch of the plurality of sub-patches is a maximally reduced patch (MRP) having an even number of edges and a first side defined by exactly one of the even number of edges, and selecting the first MST based at least in part on the first sub-patch.

Selecting the first MST may include determining whether the patch is triangular or quadrilateral, and selecting the first MST from the plurality of MSTs comprising: a first triangular MST for a triangular patch, the first triangular MST having one internal singularity; a second triangular MST for the triangular patch, the second triangular MST having two internal singularities; a first quadrilateral MST for a quadrilateral patch, the first quadrilateral MST having two internal singularities; a second quadrilateral MST for the quadrilateral patch, the second quadrilateral MST having one internal singularity; and a third quadrilateral MST for the quadrilateral patch, the third quadrilateral MST having three internal singularities.

Replacing the patch with the first MST may include: aligning each corner of a plurality of corners of the first MST with a corresponding convex corner of a plurality of convex corners of the patch; calculating a set of coordinates for each quadmesh vertex of a plurality of quadmesh vertices of the first MST; and overlaying, using the correspondence between corners and the sets of coordinates, the quadmesh of the first MST onto the polygon mesh. Obtaining the polygon mesh may include: receiving a quad-dominant mesh comprising a plurality of quadrilaterals and a plurality of non-quadrilateral elements; converting the quad-dominant mesh into a triangular mesh; matching dual edges of the triangular mesh; and forming the polygon mesh from triangles of the triangular mesh that are paired from dual edges.

In another aspect, the present disclosure provides a method for optimizing a computer graphic structure. The method includes: obtaining, by a computing device, a first polygon mesh representing the computer graphic structure, the polygon mesh comprising a plurality of polygonal faces and a plurality of vertices each located at an intersection of at least two of the faces, the plurality of vertices including a plurality of singularities; identifying a first singularity of the plurality of singularities; identifying, using the first singularity, a second polygon mesh comprising a first subset of the plurality of faces and containing the first singularity, a second singularity of the plurality of singularities, and a third singularity of the plurality of singularities; selecting, based on one or more characteristics of the second polygon mesh, a first minimum singularity template (MST) that corresponds to a third polygon mesh comprising a plurality of quadrilateral faces and having three or fewer singularities; and replacing, within the first polygon mesh, the second polygon mesh with the third polygon mesh.

Obtaining the first polygon mesh may include: generating a fourth polygon mesh for the structure, the fourth polygon mesh comprising a plurality of triangular faces; subdividing each triangular face in the fourth polygon mesh into three quadrilaterals to produce a quadmesh with no singularities; and adding one or more layers of new quadrilaterals into the quadmesh to produce the first polygon mesh. Identifying the second polygon mesh may include identifying, as the second singularity, the vertex of the plurality of vertices that is the topologically closest singularity to the first singularity. Identifying the second singularity may further include performing a Dijkstra shortest distance search from the first singularity. Identifying the second polygon mesh further includes identifying the third singularity using a breadth-first search of the plurality of vertices that begins at a first vertex disposed between the first singularity and the second singularity. Selecting the first MST may include: dividing the second polygon mesh into a plurality of sub-patches, wherein a first sub-patch of the plurality of sub-patches is a maximally reduced patch (MRP) having an even number of edges and a first side defined by exactly one of the even number of edges; and selecting the first MST based at least in part on the first sub-patch.

The foregoing and other aspects and advantages of the invention will appear from the following description. In the description, reference is made to the accompanying drawings that form a part hereof, and in which there is shown by way of illustration a preferred embodiment of the invention. Such embodiment does not necessarily represent the full scope of the invention, however, and reference is made therefore to the claims and herein for interpreting the scope of the invention.

BRIEF DESCRIPTION OF THE DRAWINGS

FIG. **1** is a diagram of an exemplary mesh optimization for a computer graphic structure in accordance with the present disclosure.

FIG. **2**A is a diagram of an exemplary three-sided patch in accordance with the present disclosure.

FIG. **2**B is a diagram of an exemplary four-sided patch in accordance with the present disclosure.

FIG. **2**C is a diagram of an exemplary five-sided patch in accordance with the present disclosure.

FIG. **3** is a flowchart of a method of identifying a patch in accordance with the present disclosure.

FIG. **4** is a diagram comparing a patch identified using an existing algorithm with a patch identified using an algorithm in accordance with the present disclosure.

FIG. **5**A is a diagram of a triangular quadmesh patch.

FIG. **5**B is a diagram of a pentagonal quadmesh patch.

FIG. **6**A is a diagram of a maximal patch reduction of a triangular patch in accordance with the present disclosure.

FIG. **6**B is a diagram of a maximal patch reduction of a quadrilateral patch in accordance with the present disclosure.

FIGS. **7**A-B are diagrams of exemplary minimum singularity templates for a triangular patch.

FIGS. **8**A-C are diagrams of exemplary minimum singularity templates for a quadrilateral patch.

FIG. **9** is a flowchart of an exemplary method for selecting a minimum singularity template in accordance with the present disclosure.

FIG. **10** is a flowchart of an exemplary method for replacing a patch of a polygon mesh with a minimum singularity template.

FIGS. **11**A-D are exemplary optimizations of a polygon mesh for a computer graphic structure.

FIG. **12** is a flowchart of an exemplary method for generating a polygon mesh in accordance with the present disclosure.

FIG. **13** is an exemplary optimization of a polygon mesh using the method of FIG. **12**.

FIG. **14** is an exemplary optimization of a quad-heavy polygon mesh in accordance with the present disclosure.

DETAILED DESCRIPTION OF THE INVENTION

Described here are systems and computer-implemented methods for modifying and generating polygonal meshes in computer graphics applications, which meshes are both geometrically and topologically optimized. Topological optimization is achieved by reducing singularities in the mesh. Generally, with reference to FIG. **1**, the user provides, or the system generates a rule for ranking singularities **104** within a quadrilateral mesh **102** of an object **100** (e.g., distance of a singularity **104** from the boundary **106**, role of singularity **104** in geometric distortion of the mesh **102**). Thereafter, these singularities **104** are sorted and incrementally removed as follows. At least three singularities **104** are needed to participate in the removal process; this leads to the idea of exploiting "patch," which is a well-known structure in mesh generation in various contexts. The present disclosure uses a patch **110** as a subset of the mesh **102** containing at least three singularities **104**A-C. Starting with three nearby singularities, additional elements (vertices, edges, surfaces, and/or polygons) are incrementally added to the patch **110**, while maintaining certain desirable properties of the patch **110**, described below.

One such property is that the boundary **112** of the patch **110** must contain at least 3 topologically convex corners **114**A-D; FIG. **1** illustrates a patch **110** with four such corners **114**A-D. Once such a patch **110** has been created, the elements within this patch **110** are replaced by a new set of elements with fewer singularities, using the Minimum Singularity Template (MST). A smoothing step is then carried out to improve the geometric quality. The process is then repeated, starting from three new singularities, until no new patches can be constructed, resulting in an optimized quad mesh **150**. While the entire process can be automated, it is also amenable to user interactions, in that specific singularities can be identified and removed at any stage.

The present methods employ the following geometric definitions, among others, for identifying singularities to be removed and constructing patches to remove the singularities.

The "valence" of a vertex v, is the number of edges incident on the vertex. A vertex with "n" valence is denoted by Vn. An internal vertex with valence **4** is considered regular, otherwise it is an irregular vertex, or "singularity." An internal vertex with valence **2** is called a doublet. In some embodiments, the present methods may be used to address V**3** and V**5** singular nodes, as all other high valence nodes can be converted into V**3** and V**5** nodes using standard atomic face open or face close operation.

A "patch" is a sub-mesh with disc topology. The topological outer angle of a vertex on the boundary of a patch (TOA(V)) is defined as the number of faces incident on the vertex that lie outside of the patch, minus two. A vertex on the boundary of a patch is a "convex corner" if its TOA is greater than or equal 1. FIGS. **2**A-C illustrate an exemplary three-sided patch **202**, four-sided patch **204**, and five-sided patch **206**, all containing singularities and demonstrating that the corresponding convex corners **212**A-C, **214**A-D, **216**A-E of each patch **202**, **204**, **206** naturally divide the boundaries **222**, **224**, **226** of each patch into "sides" **232**A-C, **234**A-D, **236**A-E of equal number to the number of convex corners.

Nevertheless, the number of quad edges on the boundary of a patch, and thus the number of quads defining the boundary, is even. This is shown by a quad mesh M, which is locally homeomorphic to an open subset of R^2, the number of faces and edges must satisfy $4F=2E_i+E_b$, where E_i is the number of internal edges shared by two faces, and E_b is the number of boundary edges shared by one face. This equation implies that E_b must be even, and confirms the existence of a quad mesh in a patch. The present disclosure provides a constructive algorithm to create an alternate quadrangulation with fewer singularities. Any quadrangulation of a patch with k convex corners will have at least |k−4| interior singularities. For example, there are 3 interior singularities **242**A-C in FIG. **2**A; all quadrangulation of the patch must contain at least one singularity.

Referring to FIG. **3**, upon receiving the user-specified or automatically generated criterion for ranking the singularities in the domain, a computer system may perform a method **300** for identifying the patches for the mesh. At step **302**, the system may determine the highest ranking singularity, which becomes the seed for the patch expansion. At step **304**, the system may grow the patch around the seed. In some embodiments, the patch expansion may be governed by a breadth-first search (BFS) using one modified heuristic: in light of the requirement for at least three singularities in a patch, the system may, instead of searching in all directions (as in classic BFS), search for one more singularity which is topologically closest to the seed using Dijkstra's shortest path algorithm or a similar algorithm. As shown in FIG. **4**, for the same mesh **400** and seed singularity **402**, this simple change creates a compact and thinner patch **420** compared to the patch **410** created from classic BFS alone.

Referring again to FIG. **3**, at step **306**, after identifying two singularities, the system may search for a third singularity using BFS starting from the vertices connecting the first two singularities. At step **308**, the boundary vertices of the patch are identified, and at step **310** their topological angles are computed determine whether any of the boundary vertices are convex corners. If there are more than five convex corners, the system may select only five of them. If convex corners are not found, at step **312** the entire patch is expanded by one layer. If the expansion reaches the mesh boundary, further expansion is halted. If the expansion reaches the mesh boundary and no corners are found based on topological angle criterion, at step **314** the system may choose the corners from the uniformly distributed boundary vertices of the patch. In some cases, this may create doublets at the corners, which must be removed immediately after remeshing the patch.

The system may then commence quadrangulation of the patch to achieve minimum singularities therein. Obtaining the least number of singularities in a patch is a constraint-satisfying linear problem, for which a solution exists only for patches which have a specific number of edges on their sides. Whereas for a quad patch, zero singularity is obtained when the number of edges on opposite sides are equal, for other polygonal patches, a linear system derived from an "interval matching" method must be solved.

The interval matching for a triangle patch leads to the system of equations of Table 1. Referring to FIG. SA, if there exists an integer solution to this system, then a triangle patch **502** can be quadrangulated with a single V3 singularity **504** inside the patch **502**. In this figure, a_0, b_0, . . . denote the number of quad edges on the patch segments. The interval matching for a pentagonal patch leads to the system of equations of Table 2. Referring to FIG. **5**B, if there exists an integer solution to this system, then a pentagonal patch **552** can be quadrangulated with a single V5 singularity **554** inside the patch **552**. Again, a_0, b_0, . . . denote the number of quad edges on the patch segments.

TABLE 1

Linear System of Equations for Triangle Patch

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{pmatrix} N_0 \\ N_1 \\ N_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{matrix} a_0 + b_0 = N_0 \\ a_1 + b_1 = N_1 \\ a_2 + b_2 = N_2 \\ a_0 - b_1 = 0 \\ a_1 - b_2 = 0 \\ a_2 - b_0 = 0 \end{matrix}$$

TABLE 2

Linear System of Equations for Pentagon Patch

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} =$$

$$\begin{pmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ N_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{matrix} a_0 + b_0 = N_0 \\ a_1 + b_1 = N_1 \\ a_2 + b_2 = N_2 \\ a_3 + b_3 = N_3 \\ a_4 + b_4 = N_4 \\ a_0 - b_3 = 0 \\ a_1 - b_4 = 0 \\ a_2 - b_0 = 0 \\ a_3 - b_1 = 0 \\ a_4 - b_2 = 0 \end{matrix}$$

If solving either system of linear equations leads to integer values to all the a_i and b_i, then the patch can be remeshed with only one singularity in the interior (and vice versa). The inversions of these matrices are pre-calculated; the main objective becomes finding the minimum number of singularities within a patch. An exhaustive search is infeasible except for small patch sizes. Therefore, the system may first divide a given patch into sub-patches. One of the sub-patches, which is referred to herein as the Maximally Reduced Patch (MRP), has one or more sides with only one edge, and all other sub-patches are 4-sided (i.e., quadrilateral) with perfect quadmeshes (i.e., no singularity).

Referring to FIG. **6**A, to find the MRP for a triangle patch $P=P(N_0,N_1,N_2)$, where N_k denotes the number of quad edges on the kth side, the system may rearrange the triangle so that $N_0 \geq N_1 \geq N_2$. Then, the patch P_0 is divided into sub-patches P_1 and P_2 according to the system:

$$m_1=\min(N_0,N_1)-1$$

$$P_0(N_0,N_1,N_2)=P_1(m_1,N_2,m_1,N_2)+P_2(N_0-m_1,1,N_2).$$

The sub-patch P_1 leads to a perfect quad mesh, and the second side of the sub-patch P_2 has only one edge. Sub-patch P_2 may be further reduced into sub-patches P_3 and P_4:

$$m_2=\min(N_0-m_1,N_2)-1$$

$$P_2(N_0-m_1,1,N_2)=P_3(m_2,1,m_2,1)+P_4(N_0-m_1-m_2,1,1).$$

With the second decomposition, the sub-patch P_3 will lead to a perfect quadmesh and P_4 has only edge on its second and third side. This forms the MRP for a given triangular P_0 patch. With these decompositions, finding the minimum number of singularities in the P_0 patch is reduced to finding minimum number of singularities in the sub-patch P_4. Sub-patch P_4 has an even number of edges on its boundary, and thus is a quadrilateral patch that can be quadrangulated as discussed below with respect to FIG. **6**B.

Referring to FIG. **6**B, to find the MRP for a quadrilateral patch $P=P(N_0,N_1,N_2,N_3)$, where N_k denotes the number of quad edges on the kth side, the system may rearrange the patch so that $N_0 \geq N_2$ and $N_1 \geq N_3$. Then, the patch P_0 is decomposed in two directions. The first decomposition is done with a vertical cut to provide two sub-patches P**1** and P**2**, wherein P**1** contains a perfect quadmesh, but the patch P**2** has only one edge on its top side:

$$m_1=\min(N_0,N_2)-1$$

$$P_0(N_0,N_1,N_2,N_3)=P_1(m_1,N_3,m_1,N_3)+P_2(N_0-m_1,N_1,1,N_3).$$

The second decomposition may be performed with a horizontal cut on P**2** to produce two sub-patches P**3** and P**4**. The sub-patch P**3** may contain a perfect quadmesh and the patch P**4** has its left and top sides each containing one edge. This forms the MRP of the original quad patch P**0**:

$$m_2=\min(N_1,N_3)-1$$

$$P_2(N_0-m_1,N_1,1,N_3)=P_3(N_0-m_1,m_2,N_0-m_1,m_2)+P_4(N_0-m_1,N_2-m_2,1,1).$$

With these two decompositions, finding the minimum singularities in the original patch P**0** is reduced to finding the minimum singularities in the patch P**4**. The quadrangubility of P**4** is guaranteed, as it contains an even number of edges. It is further noted that a pentagon can be decomposed into a quad and a triangle, or three triangles in various ways. Each triangle and quad patch can be quadrangulated with the methods described above.

Once any triangular, quadrilateral, or pentagonal patch is decomposed into sub-patches to find the MRP, one or more minimum singularity templates (MSTs) may be selected to correspond to the MRP. Referring to FIGS. **7**A-B, there are two MSTs for a triangular patch, having at most two singularities: a first MST **700**, referred to in this text as a T_1 type, has only one singularity **702**; a second MST **710**, referred to herein as a T_2 type, has only two singularities **712**, **714**. Referring to FIGS. **8**A-C, there are three MSTs for a quadrilateral patch, having at most three singularities: a first MST **800**, referred to in this text as a Q_1 type, has only one singularity **802**; a second MST **810**, referred to in this text as a Q_2 type, has one singularity **812**; and, a third MST **820**, referred to in this text as a Q_3 type, has three singularities **822**, **824**, **826**. Using these MSTs, a pentagon patch can be quadrangulated with four singularities.

Referring to FIG. **9**, given a patch with 3, 4, or 5 sides, and a certain number of quad edges on each side, the system may perform a method **900** for automatically selecting a MST to represent the patch. At step **902**, the system may check whether either system of equations for triangular or quadrilateral MRP gives an integer solution. If so, then the patch is quadrangulable with only one singularity and at step **904** the system may select either T_1 or Q_2 depending on the patch shape. If there is no integer solution, the system proceeds based on whether the patch is triangular or quadrilateral. If the patch is triangular, at step **906** the system may check the number of vertices in the sub-patch P_4. If there are three vertices on its first side, at step **908** the system may select the T_1 template; otherwise, at step **910** the system may select the T_2 template.

If the patch is quadrilateral, at step **916** the system may determine whether the patch's sides N_1,N_2,N_3 all have one edge; if so, at step **918** the system may select the Q_1 patch. If the number of edges on side N_0 and N_1 is equal and greater than one, at step **920** the system may select the Q_2 patch. If the number of edges on side N_0 and N_1 differ, at step **922** the system may select the Q_3 patch. If the patch is pentagonal, the system may decompose it into either quad-triangles pairs or in three triangular patches. Since there are many ways to decompose a pentagon, the system may check each decomposition for the number of singularities and accept the one which gives the least number of singularities.

While the system can perform the combinatorial, algebraic steps above to resolve singularities, the final patch replacement may include geometric considerations as well. The shape of a patch could be arbitrarily complex (although it is always a topological disk); therefore, the mapping from template domain to physical domain can produce large distortions and inverted elements. FIG. **10** illustrates a method **1000** that the system may perform to replace a patch with the template mesh. At step **1002**, the system may first create a correspondence between corners of the template patch with the convex corners of the patch to be replaced. That is, the system may associate each template corner with a corresponding convex corner. At step **1004**, the system may calculate a set of coordinates, such as Floater's Mean Value Coordinates, for each vertex of the template mesh with respect to its corners; therefore, each vertex of the template mesh will have coordinates in n-dimensional space, where n is the number of convex corners.

At step **1006**, the system may overlay the coordinates of the corners of the template mesh onto the coordinates of the identified patch and calculate the new coordinates of each vertex of the template mesh. At step **1008**, the system may apply a constraint (i.e. Dirichlet boundary condition) on the boundary vertices of the template mesh using the corresponding boundary vertices on the physical patch. At step **1010**, the system may optimize the patch using, e.g., Lloyds relaxation and improve the shape of quad elements, such as by using Mesquite software.

At step **1012**, the system may check for inversion of each face in the template mesh. If there is an inverted face (i.e., the face has a negative Jacobian), at step **1014** the system may apply locally injective mapping (e.g., of Schueller, etc,) to obtain a fold-free mesh. At step **1016**, if all the faces have positive Jacobian, the system may refine the mesh quality, using Mesquite or another suitable mesh node-moving software program to replace the identified patch with the template mesh, update the mesh data structures, and further integrate the selected MST into the optimized mesh.

FIGS. **11**A-D illustrate the potential results of performing the MST method on existing quad-meshes: in FIG. **11**A, the Abaqus program was used to generate the quad-mesh **1102**, and the MST method modified the quad-mesh **1102** to produce an optimized mesh **1104**; in FIG. **11**B, the Gmsh program was used to generate the quad-mesh **1112**, and the MST method modified the quad-mesh **1112** to produce an optimized mesh **1114**; in FIG. **11**C, the Gmsh program was used to generate the quad-mesh **1122**, and the MST method modified the quad-mesh **1122** to produce an optimized mesh **1124**; and, in FIG. **11**D, the Cubit program was used to generate the quad-mesh **1132**, and the MST method modified the quad-mesh **1132** to produce an optimized mesh **1134**. Table 3 lists the comparative results. In all cases, the number of quad faces remains almost constant, while at least 82% of singularities are removed. Although in all the experiments, the final mesh quality is within acceptable range (as defined by the Verdict Software), there are two ways to improve the results: (1) allow selective refinement near high aspect ratio elements, and (2) use L_∞ shape optimization methods instead of average quality improvement methods.

TABLE 3

Mesh qualities using Verdict software; (x/y) denotes value of input mesh and output mesh respectively.

	Square hole	Dolphin	Bird	5-holes
Source	Abaqus	Gmsh	Gmsh	Cubit
Faces	4250/4015	5856/5586	2399/2271	9923/10085
Singularities	156/12 (92%)	474/38 (92%)	272/47 (82%)	627/70 (89%)
# Iterations	17	30	20	40
Aspect Ratio (L)	1.70/1.54	2.60/3.65	2.24/3.44	1.71/3.04
Condition number (L)	1.65/1.32	3.18/5.30	3.42/3.54	1.81/1.97
Distortion (L)	0.40/0.60	0.27/0.30	0.38/0.41	0.46/0.45
MinAngle (L)	48/57	32/24	25/22	46/37
MaxAngle (H)	140/126	155/162	158/159	140/145
Oddy (L)	3.39/1.5	18.31/54	21.46/23	4.56/5.78
Scaled Jacobian (H)	0.65/0.80	0.42/0.30	0.35/0.34	0.63/0.57
Skew (L)	0.58/0.40	0.74/0.86	0.83/0.88	0.66/0.68
Stretch (L)	1.04/1.26	15.03/23.2	12.81/18.0	1.45/2.06
Taper (L)	0.47/0.30	0.88/0.56	0.64/0.73	0.43/0.39

Besides improving the quality of an existing quad mesh, this disclosure provides additional processes that exploit the above MST operations to (1) generate an ab-initio high-quality quadrilateral mesh in a complex domain, and (2) convert a quad-dominant mesh to a pure quadrilateral mesh. The first process is referred to herein as combinatorial quadrilateral mesh generation (CQM). FIG. **12** illustrates an exemplary CQM method **1200** that generates a mesh for a complex structure as described above. At step **1202**, the system may generate a triangle mesh for the structure. Answers: The triangle mesh is a representation of the whole structure, and also functions as a seed mesh around which layers of quadrilateral elements are constructed as described below. There are several triangle mesh generators that are available; any of them can be used to generate the triangle mesh. The parameters of the seed mesh include, without limitation, the number of elements desired (at the final stage): the greater the number of elements desired, the finer is the mesh. At step **1204**, the system may subdivide each triangle in the triangular mesh into three quadrilaterals in accordance with the presently-described algorithms.

At step **1206**, the system may iteratively add layers of quadrilaterals at the boundaries of the mesh and integrate the new quadrilaterals into the mesh. These new layers exhibit ideal topological quality but require geometric optimization. These layers are added incrementally to control the maximum distortion in the final mesh. Experimentation has shown that typically 5-6 layers are sufficient; adding more layers will minimize distortion, but for thin regions, adding too many layers may not be possible. The layers push the quadrilateral elements of the initial mesh, which may have singularities (i.e., are not topologically ideal) inside the domain, allowing aggressive MST operations. At step **1208**, such MST operations are applied to interior elements to reduce singularities in the mesh. If the MST operations result in a mesh with high distortion or if the mesh is coarse, new layers of quadrilaterals are inserted by returning to the previous step.

The CQM process **1200** of the present disclosure works with a quadrilateral mesh and only improves the existing topology of a quadmesh using MST operations. At every stage of the CQM process **1200**, there is a topologically valid all-quadrilateral mesh. Thus, the parity (i.e., number of edges) of the boundary is always even. Additionally, CQM can exploit parallelism that exists in non-overlapping patches. In general, in the beginning, there are a large number of singularities in a mesh which are topologically far away, and the system can generate many independent patches. However, as the singularity reduction progresses, patches start overlapping and the scope of parallelism decreases. FIG. **13** provides a procedural illustration of the CQM process: a base triangular mesh **1300** is generated; each triangle is broken into three quads to form an initial quad-mesh **1302**; a pillow function is performed to produce a first optimized mesh **1304** from the initial quad-mesh **1302**; and, MST operations are performed to produce the final mesh **1306**.

According to the present disclosure, the system may also improve a quad-dominant mesh. A quad-dominant mesh contains a small number of non-quadrilateral elements (usually triangles and pentagons). In general, algorithms to generate quad-dominant meshes are much simpler and efficient than generating a pure quadmesh. However, the common techniques create large numbers of singularities and may add new operations, such as boundary splitting, which reduce the efficiency. The present disclosure includes a "far triangles agglomeration" (FTA) method that overcomes the drawbacks. In the FTA method, triangles are paired using Edmonds' Perfect Graph Matching algorithm as follows: (1) convert the quad-dominant mesh into a pure triangular mesh by dividing every quad element into two triangles, (2) construct a dual graph of the triangular mesh, (3) apply Edmonds' algorithm to match dual edges, and (4) form a new quadrilateral mesh from the triangles paired from dual edges. Once a pure quadrilateral mesh is obtained, the system may apply MST operations to reduce singularities in the mesh. FIG. **14** shows an example of a quad-dominant meshes **1402** with less than 10% triangle elements (using greedy triangles matching in triangular meshes). After applying Edmonds algorithm, these meshes are converted into a mesh **1404** of pure quads. The pure quad mesh contains a large number of singularities, so MST operations are applied to produce the improved mesh **1406**.

The present invention has been described in terms of one or more preferred embodiments, and it should be appreciated that many equivalents, alternatives, variations, and modifications, aside from those expressly stated, are possible and within the scope of the invention.

The invention claimed is:

1. A method of optimizing a computer graphic structure, the method comprising:

receiving a polygon mesh representing the computer graphic structure, the polygon mesh comprising a plurality of polygonal faces and a plurality of vertices each located at an intersection of at least two of the faces, the plurality of vertices including a plurality of singularities at which more or less than four of the faces intersect;

receiving a criterion for ranking the plurality of singularities according to a geometric quality of the computer graphic structure;

determining, using the criterion, that a first singularity of the plurality of singularities is the highest ranking singularity;

identifying, using a shortest path algorithm, that a second singularity of the plurality of singularities is topologically closest, of the plurality of singularities within the polygon mesh, to the first singularity;

identifying, using a breadth-first search of the plurality of vertices that begins at a first vertex disposed between the first singularity and the second singularity, a third singularity of the plurality of singularities;

determining a patch of the polygon mesh, the patch containing the first singularity, the second singularity, and the third singularity and having a first boundary comprising:

a plurality of sides each defined by one or more of a plurality of edges between two of the plurality of faces; and

a plurality of convex corners each defined by a corresponding vertex of the plurality of vertices, each of the convex corners located at the intersection of two of the plurality of sides;

determining a first minimum number of singularities that must be located within the patch;

selecting, based on the first minimum number of singularities, a first minimum singularity template (MST) of a plurality of MSTs each representing a corresponding quadmesh; and

replacing, within the polygon mesh, the patch with the first MST to produce an optimized polygon mesh representing the computer graphic structure with the geometric quality, the optimized polygon mesh comprising fewer than the plurality of singularities.

2. The method of claim 1, wherein determining the first minimum number of singularities for the patch comprises:

dividing the patch into a plurality of sub-patches, wherein a first sub-patch of the plurality of sub-patches is a maximally reduced patch (MRP) having a second boundary with an even number of the plurality of edges and a first MRP side defined by exactly one of the plurality of edges, and each other of the plurality of sub-patches are quadrilateral and each contain a corresponding quadmesh that has no singularities; and

determining, as the first minimum number of singularities for the patch, a second minimum number of singularities that must be located within the first sub-patch.

3. The method of claim 1, wherein selecting the first MST comprises determining that the patch is quadrangulable with only one singularity.

4. The method of claim 1, wherein selecting the first MST comprises:

determining whether the patch is triangular or quadrilateral; and

selecting the first MST from the plurality of MSTs comprising:

a first triangular MST for a triangular patch, the first triangular MST having one internal singularity;

a second triangular MST for the triangular patch, the second triangular MST having two internal singularities;

a first quadrilateral MST for a quadrilateral patch, the first quadrilateral MST having two internal singularities;

a second quadrilateral MST for the quadrilateral patch, the second quadrilateral MST having one internal singularity; and

a third quadrilateral MST for the quadrilateral patch, the third quadrilateral MST having three internal singularities.

5. The method of claim 1, wherein replacing the patch with the first MST comprises:

creating a correspondence between each corner of a plurality of corners of the MST with a corresponding convex corner of the plurality of convex corners of the patch;

calculating a set of coordinates for each quadmesh vertex of a plurality of quadmesh vertices of the first MST;

overlaying, using the correspondence between corners and the sets of coordinates, the quadmesh of the first MST onto the polygon mesh; and

applying a locally injective mapping to each inverted face of a plurality of faces of the quadmesh to produce a fold-free mesh in which each of the plurality of faces has a positive Jacobian.

6. A method comprising:

obtaining a polygon mesh representing a computer graphic structure, the polygon mesh comprising a plurality of polygonal faces and a plurality of vertices each located at an intersection of at least two of the faces, wherein some of the plurality of vertices are singularities;

determining, based on a first singularity of the plurality of vertices, a patch of the polygon mesh, the patch containing the first singularity and having a first boundary;

selecting, based on one or more characteristics of the patch, a first minimum singularity template (MST) of a plurality of MSTs each representing a corresponding quadmesh that has three or fewer singularities; and

replacing, within the polygon mesh, the patch with the first MST to produce an optimized polygon mesh representing the computer graphic structure, the optimized polygon mesh comprising fewer singularities than the polygon mesh;

wherein determining the patch comprises:

identifying, as a second singularity, the vertex of the plurality of vertices that is the topologically closest singularity to the first singularity, the patch containing the second singularity; and

identifying, using a breadth-first search of the plurality of vertices that begins at a first vertex disposed between the first singularity and the second singularity, a third singularity of the plurality of vertices, the patch containing the third singularity.

7. The method of claim **6**, wherein obtaining the polygon mesh comprises:

generating a triangular mesh for the structure;

subdividing each triangle in the triangular mesh into three quadrilaterals to produce a quadmesh with no singularities; and

adding one or more layers of new quadrilaterals into the quadmesh, thus producing the vertices that are singularities.

8. The method of claim **6**, wherein selecting the first MST comprises:

dividing the patch into a plurality of sub-patches, wherein a first sub-patch of the plurality of sub-patches is a maximally reduced patch (MRP) having an even number of edges and a first side defined by exactly one of the even number of edges; and

selecting the first MST based at least in part on the first sub-patch.

9. The method of claim **6**, wherein replacing the patch with the first MST comprises:

aligning each corner of a plurality of corners of the first MST with a corresponding convex corner of a plurality of convex corners of the patch;

calculating a set of coordinates for each quadmesh vertex of a plurality of quadmesh vertices of the first MST; and

overlaying, using the correspondence between corners and the sets of coordinates, the quadmesh of the first MST onto the polygon mesh.

10. The method of claim **6**, wherein obtaining the polygon mesh comprises:

receiving a quad-dominant mesh comprising a plurality of quadrilaterals and a plurality of non-quadrilateral elements;

converting the quad-dominant mesh into a triangular mesh;

matching dual edges of the triangular mesh; and

forming the polygon mesh from triangles of the triangular mesh that are paired from dual edges.

11. The method of claim **6**, wherein identifying the second singularity comprises performing a Dijkstra shortest distance search from the first singularity.

12. The method of claim **6**, wherein obtaining the polygon mesh comprises:

generating a triangular mesh for the structure;

subdividing each triangle in the triangular mesh into three quadrilaterals to produce a quadmesh with no singularities; and

adding one or more layers of new quadrilaterals into the quadmesh, thus producing the vertices that are singularities.

13. A method comprising:

obtaining a polygon mesh representing a computer graphic structure, the polygon mesh comprising a plurality of polygonal faces and a plurality of vertices each located at an intersection of at least two of the faces, wherein some of the plurality of vertices are singularities;

determining, based on a first singularity of the plurality of vertices, a patch of the polygon mesh, the patch containing the first singularity and having a first boundary;

selecting, based on one or more characteristics of the patch, a first minimum singularity template (MST) of a plurality of MSTs each representing a corresponding quadmesh that has three or fewer singularities; and

replacing, within the polygon mesh, the patch with the first MST to produce an optimized polygon mesh representing the computer graphic structure, the optimized polygon mesh comprising fewer singularities than the polygon mesh;

wherein selecting the first MST comprises:

determining whether the patch is triangular or quadrilateral; and

selecting the first MST from the plurality of MSTs comprising:

a first triangular MST for a triangular patch, the first triangular MST having one internal singularity;

a second triangular MST for the triangular patch, the second triangular MST having two internal singularities;

a first quadrilateral MST for a quadrilateral patch, the first quadrilateral MST having two internal singularities;

a second quadrilateral MST for the quadrilateral patch, the second quadrilateral MST having one internal singularity; and

a third quadrilateral MST for the quadrilateral patch, the third quadrilateral MST having three internal singularities.

14. The method of claim **13**, wherein determining the patch comprises identifying, as a second singularity, the vertex of the plurality of vertices that is the topologically closest singularity to the first singularity, the patch containing the second singularity.

15. The method of claim **14**, wherein identifying the second singularity comprises performing a Dijkstra shortest distance search from the first singularity.

16. The method of claim **13**, wherein obtaining the polygon mesh comprises:

receiving a quad-dominant mesh comprising a plurality of quadrilaterals and a plurality of non-quadrilateral elements;

converting the quad-dominant mesh into a triangular mesh;

matching dual edges of the triangular mesh; and

forming the polygon mesh from triangles of the triangular mesh that are paired from dual edges.

17. A method for optimizing a computer graphic structure, the method comprising:

obtaining, by a computing device, a first polygon mesh representing the computer graphic structure, the polygon mesh comprising a plurality of polygonal faces and a plurality of vertices each located at an intersection of at least two of the faces, the plurality of vertices including a plurality of singularities;

identifying a first singularity of the plurality of singularities;

identifying, using the first singularity, a second polygon mesh comprising a first subset of the plurality of faces and containing the first singularity, a second singularity of the plurality of singularities, and a third singularity of the plurality of singularities;

selecting, based on one or more characteristics of the second polygon mesh, a first minimum singularity template (MST) that corresponds to a third polygon mesh comprising a plurality of quadrilateral faces and having three or fewer singularities; and

replacing, within the first polygon mesh, the second polygon mesh with the third polygon mesh to produce an optimized polygon mesh representing the computer graphic structure, the optimized polygon mesh comprising fewer than the plurality of singularities wherein identifying the second polygon mesh comprises:

identifying, as the second singularity, the vertex of the plurality of vertices that is the topologically closest singularity to the first singularity; and

identifying the third singularity using a breadth-first search of the plurality of vertices that begins at a first vertex disposed between the first singularity and the second singularity.

18. The method of claim **17**, wherein obtaining the first polygon mesh comprises:

generating a fourth polygon mesh for the structure, the fourth polygon mesh comprising a plurality of triangular faces;

subdividing each triangular face in the fourth polygon mesh into three quadrilaterals to produce a quadmesh with no singularities; and

adding one or more layers of new quadrilaterals into the quadmesh to produce the first polygon mesh.

19. The method of claim **17**, wherein identifying the second singularity comprises performing a Dijkstra shortest distance search from the first singularity.

20. The method of claim **17**, wherein selecting the first MST comprises:

dividing the second polygon mesh into a plurality of sub-patches, wherein a first sub-patch of the plurality of sub-patches is a maximally reduced patch (MRP) having an even number of edges and a first side defined by exactly one of the even number of edges; and

selecting the first MST based at least in part on the first sub-patch.

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